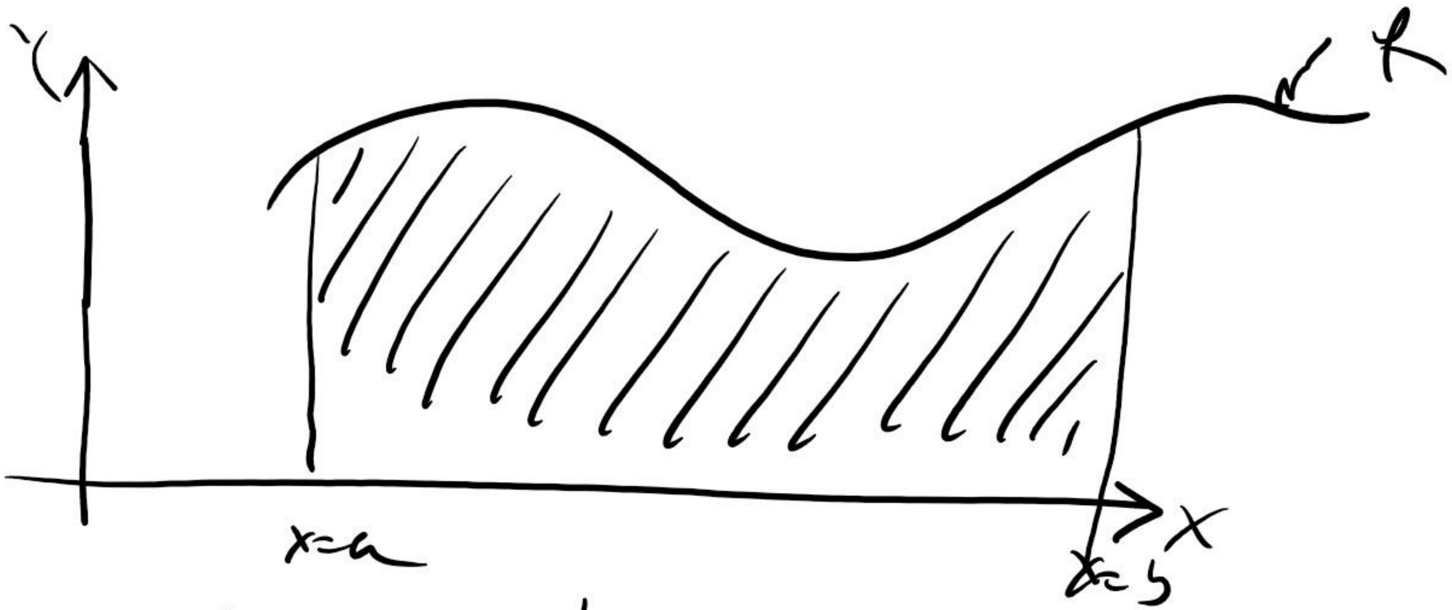
ÚLTIMA AULA: A PRETEXTO DE CALCULAR ÁREAS DE FIGURAS PLANAS, DEFINIMOS INTEGRAL.

<u>OBS</u> AS FIGURAS PLANAS EM QUESTÃO SÃO DADAS DA SEGUINTE MANEIRA:

HÁ UMA FUNÇÃO f, DEFINIDA NUM INTERVALO

$$f : [a,b] \longrightarrow \mathbb{R} \quad \geq 0 \ \wedge \ \text{LIMITADA}$$

A NOSSA FIGURA PLANA É DELIMITADA PELO GRÁFICO DE f, O EIXO DOS xx e RECTAS VERTICAIS $x=a$ $\wedge$ $x=b$.

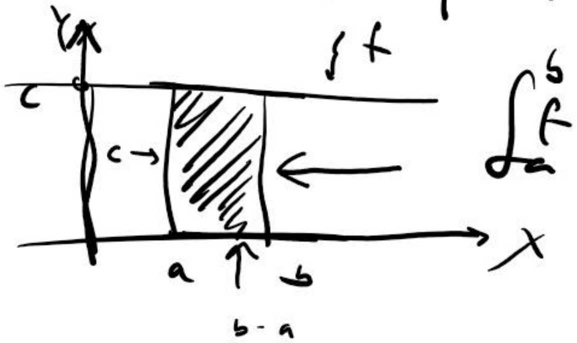


NOTAÇÃO: $\int_a^b f$, $\int_a^b f(x)\,dx$, $\int_a^b f(s)\,ds$

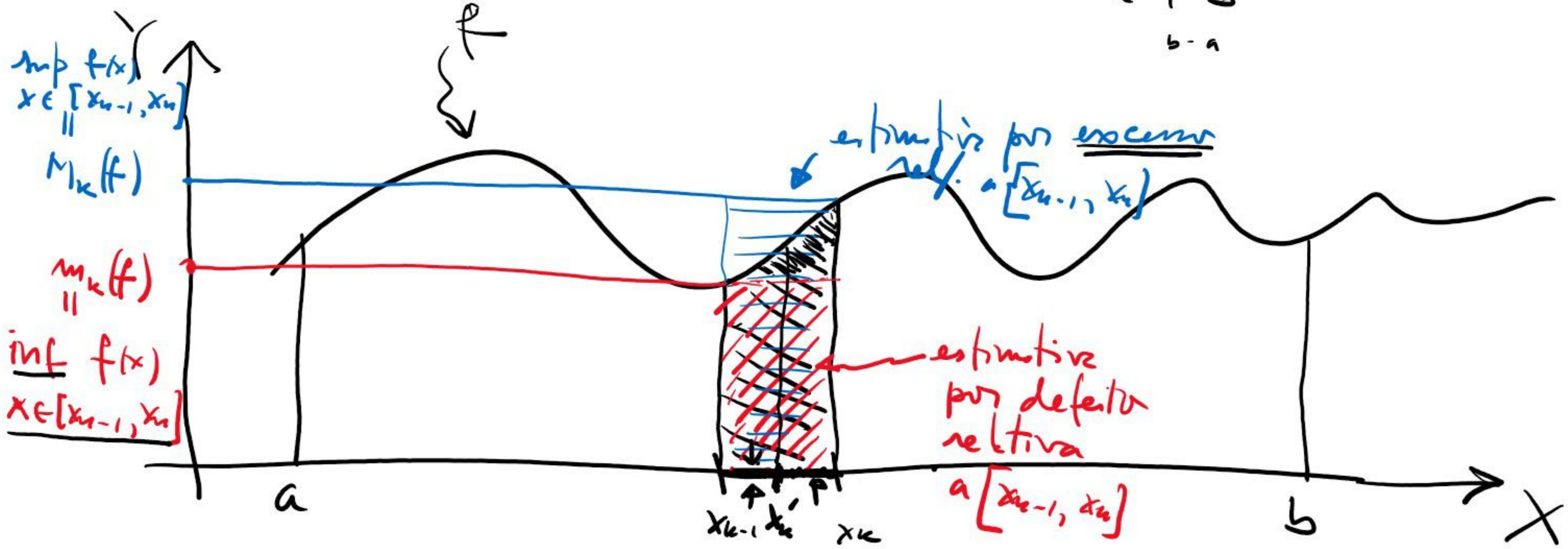
Variável de integração é muda (DUMMY)
Tal como o índice de soma no somatório

PORTANTO $\int_a^b f$ é a área pretendida da figura plana em causa.

Exemplo (i) FUNÇÃO CONSTANTE:



$$\int_a^b f = \underline{\underline{c(b-a)}}$$



$$d = \{a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k < \cdots < x_{n-1} < x_n = b\}$$

$$s_d(f) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}) \leq \text{área pretendida} \leq \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1}) = S_d(f)$$

SOMA DE DARBOUX INFERIOR DE f REL/ À DECOMP. d

SOMA DE DARBOUX SUPERIOR DE f REL/ À DECOMP. d

$$d' = d \cup \{x'_k\} \implies$$

$$s_d(f) \leq s_{d'}(f) \leq \underline{\int_a^b} f \leq \text{área pretendida} \leq \overline{\int_a^b} f \leq S_{d'}(f) \leq S_d(f)$$

$$\underline{\int_a^b} f = \underline{\sup} \left\{ \overline{\underline{s_d(f) : d \text{ é decomp. de } [a,b]}} \right\} \quad \Bigg| \quad \inf \left\{ \overline{S_d(f) : d \text{ é decomp. de } [a,b]} \right\} = \overline{\int_a^b} f$$

$$\underline{\int_a^b} f = \sup\left\{ \underline{\overline{s_d(f)} : d \text{ é decomp. de } [a,b]} \right\} \quad \Big| \quad \inf\left\{ \overline{S_d(f) : d \text{ é decomp. de } [a,b]} \right\} = \overline{\int_a^b} f$$

há uma ∞ de decomp.

DEF. f é integrável em $[a,b] \overset{def.}{\Longleftrightarrow}$

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

Obs. Nestas circunstâncias a nossa pretendida é esse valor comum:

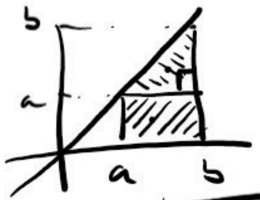
$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$$

Exemplos: dada decomp qq d:

(i) funções constantes: $s_d(f) = c(b-a) = S_d(f)$

$$\Rightarrow \underline{\int_a^b} f = c(b-a) = \overline{\int_a^b} f \Rightarrow \boxed{\int_a^b f = c(b-a)}$$

(ii) função identidade: $\int_a^b f = \frac{b^2-a^2}{2}$??



$$\boxed{s_d(f) = b^2-a^2-S_d(f)} \rightsquigarrow \underline{\int_a^b} f = b^2-a^2-\overline{\int_a^b} f \rightarrow \boxed{\int_a^b f = b^2-a^2-\int_a^b f}$$

$$2\int_a^b f = b^2-a^2$$

$$\boxed{\int_a^b f = \frac{b^2-a^2}{2}}$$

(iii) função de DIRICHLET: D $\quad s_d(D) = 0 \quad S_d(D) = b-a$

$$\Downarrow$$

$$\underline{\int_a^b} D = 0 \qquad \overline{\int_a^b} D = b-a$$

portanto D não é integrável.

PROP

portanto D não é integrável. $\int_0 F = \frac{\ }{2}$

PROP Se f é contínua em $[a,b]$ então f é integrável em $[a,b]$

REGRA DE BARROW

There is no free lunch

Seja f integrável em $[a,b]$

Seja F tal que $F'(x) = f(x) \quad \forall x \in [a,b]$

Então $\int_a^b f = F(b) - F(a)$.

DEF. F é a PRIMITIVA de f

TÓPICO: TÉCNICAS DE PRIMITIVAÇÃO:

(i) PRIMITIVAS IMEDIATAS

(ii) PRIMITIVAÇÃO POR PARTES $\left((fg)' = f'g + fg' \right)$

(iii) PRIMITIVAÇÃO POR SUBSTITUIÇÃO
("TROCAMOS" A VARIÁVEL DE INTEGRAÇÃO
POR UMA NOVA VARIÁVEL

PROP

Seja F_0 primitiva de f em $]a,b[$
(i.e. $F_0'(x) = f(x) \quad \forall x \in]a,b[$)

Seja F outra primitiva de f em $]a,b[$
(i.e. $F'(x) = f(x) \quad \forall x \in]a,b[$)

Então $F(x) = F_0(x) + c \quad \forall x \in]a,b[$

onde $\underline{c}$ é uma constante.

<u>dem</u> Considere $G(x) = F(x) - F_0(x) \quad \forall x \in]a,b[$

$$\underline{G'(x) = F'(x) - F_0'(x) = f(x) - f(x) = 0} \quad \forall x \in]a,b[$$

Seja $a_0 \in]a,b[$ e $x \in]a,b[\setminus \{a_0\}$; aplicamos LAGRANGE a G e a $\underline{a_0}$ e $\underline{x}$

$$\frac{G(x) - G(a_0)}{x - a_0} = G'(c) = 0$$

donde $G(x) = G(a_0) \quad \forall x \in]a,b[\setminus \{a_0\}$
constante

donde G é constante; mas $G(x) = F(x) - F_0(x)$, donde

$F(x) = F_0(x) + cte. \quad \forall x \in]c, b[$

<u>Ejemplos</u>

(i) Sea $a \in \mathbb{R}$ constante.

$$(ax)' = a \quad \text{donde}$$

$$P(a) = ax + c$$

(ii) con $\boxed{\alpha \neq -1}$

$$\left(\frac{x^{\alpha+1}}{\alpha+1}\right)' = \frac{\alpha+1}{\alpha+1} x^{\alpha+1-1} = x^{\alpha}$$

$$\Longrightarrow P x^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + c$$

(iii) $\alpha = -1$ $\quad (\log|x|)' = \frac{1}{x} = x^{-1} \Longrightarrow$

$$P\,\frac{1}{x} = \log|x| + c$$

PROP A PRIMITIVAÇÃO É OPERAÇÃO LINEAR.

dem É uma consequência do facto que a derivada é operação linear ∎

Exemplo:

$$P(1+3x+x^2) = P1 + 3Px^1 + Px^2 = x + 3\,\frac{1}{1+1}x^{1+1} + \frac{1}{2+1}x^{2+1} + c$$
$$= x + \frac{3}{2}x^2 + \frac{1}{3}x^3 + c$$

LISTA DE PRIMITIVAS IMEDIATAS $\qquad u = u(x)$

- $P\, u' u^{\alpha} = \frac{1}{\alpha+1} u^{\alpha+1} + c \quad \forall\ \alpha \neq -1$
- $P \frac{u'}{u} = \log|u| + c$
- $P\, u' e^{u} = e^{u} + c$
- $P\, u' \sin u = -\cos u + c$
- $P\, u' \cos u = \sin u + c$
- $P \frac{u'}{1+u^2} = \operatorname{arctg} u + c$
- $P \frac{u'}{\sqrt{1-u^2}} = \arcsin u + c$
- $P \frac{u'}{\sqrt{u^2+1}} = \log\left(u + \sqrt{u^2+1}\right) + c$
- $P \frac{u'}{\sqrt{u^2-1}} = \log\left|u + \sqrt{u^2-1}\right| + c$
- $P\, u' \sec^2 u = \operatorname{tg} u + c$
- $P\, u' \operatorname{cosec}^2 u = -\operatorname{cotg} u + c$
- $P\, u' \sec u \operatorname{tg} u = \sec u + c$
- $P\, u' \operatorname{cosec} u \operatorname{cotg} u = -\operatorname{cosec} u + c$

$$\boxed{(\operatorname{arctg} t)' = \frac{1}{1+t^2}}$$

$$\boxed{(\arcsin t)' = \frac{1}{\sqrt{1-t^2}}}$$

$$\boxed{\begin{aligned} \sec t &= \frac{1}{\cos t} \\ \operatorname{cosec} t &= \frac{1}{\sin t} \end{aligned}}$$

Exemplo

$$P\, x(x^2+1)^{999} \overset{\checkmark}{=} \frac{1}{2} P(2x)(x^2+1)^{999} = \frac{1}{2} P\, \underbrace{(2x)(x^2+1)^{999}}_{u' \cdot u^{\alpha}} \qquad c/\ \alpha \neq -1$$

$$= \frac{1}{2}\, \frac{1}{999+1} (x^2+1)^{999+1} + c = \frac{1}{2000} (x^2+1)^{1000} + c$$

$$P\, \frac{1}{1+4x^2} = P\, \frac{1}{1+(2x)^2} \overset{\checkmark}{=} \frac{1}{2} P\, \frac{2}{1+(2x)^2} = \frac{1}{2} \operatorname{arctg}(2x) + c$$

$u \Rightarrow u' = 2$

$$P\, \frac{x}{1+4x^2} = \frac{1}{8} P\, \frac{8x}{1+4x^2} = \frac{1}{8} P\, \frac{(1+4x^2)'}{1+4x^2} = \frac{1}{8} \log(1+4x^2) + c$$

$$P \tan x = P\, \frac{\sin x}{\cos x} = (-1)\, P\, \frac{-\sin x}{\cos x} = -P\, \frac{(\cos x)'}{\cos x} = -\log|\cos x| + c$$

$$\int \frac{\log x}{x} = \int \frac{1}{x} \log x = \int \underbrace{(\log x)'}_{u'} \cdot \underbrace{\log x}_{u^1} = \frac{1}{1+1} (\log x)^{1+1} + c = \frac{1}{2} \log^2 x + c$$

$$\int \sqrt{\frac{1+x}{1-x}} = \int \sqrt{\frac{(1+x)(1+x)}{(1-x)(1+x)}} = \int \sqrt{\frac{(1+x)^2}{1-x^2}} = \int \frac{1+x}{\sqrt{1-x^2}} =$$

$$= \int \frac{1}{\sqrt{1-x^2}} + \int \frac{x}{\sqrt{1-x^2}} = \arcsin x + \int x (\underbrace{1-x^2}_{u})^{-\frac{1}{2}} \quad \alpha \qquad \Rightarrow u' = (-2)x$$

$$= \arcsin x - \frac{1}{2} \int \underbrace{(-2x)}_{u'} \cdot \underbrace{(1-x^2)^{-\frac{1}{2}}}_{u^\alpha} = \arcsin x - \frac{1}{\cancel{2}} \frac{1}{\cancel{-\frac{1}{2}+1}} (1-x^2)^{-\frac{1}{2}+1} + c$$

$$= \arcsin x - \sqrt{1-x^2} + c$$

$$\int \sec x = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} = \int \frac{\sec^2 x + \sec x \, \mathrm{tg}\, x}{\sec x + \mathrm{tg}\, x} = \log|\sec x + \mathrm{tg}\, x| + c$$

$$\int \mathrm{cosec}\, x = \int \mathrm{cosec}\, x \, \frac{\mathrm{cosec}\, x + \mathrm{cotg}\, x}{\mathrm{cosec}\, x + \mathrm{cotg}\, x} = \int \frac{\mathrm{cosec}^2 x + \mathrm{cosec}\, x \, \mathrm{cotg}\, x}{\mathrm{cosec}\, x + \mathrm{cotg}\, x} = -\log|\mathrm{cosec}\, x + \mathrm{cotg}\, x| + c$$

$$\int \sin x \cos x = \int \cos x \sin x = \int \underbrace{(\sin x)' \sin x}_{u' \cdot u} = \frac{1}{1+1}(\sin x)^{1+1} + C = \frac{1}{2}\sin^2 x + C$$

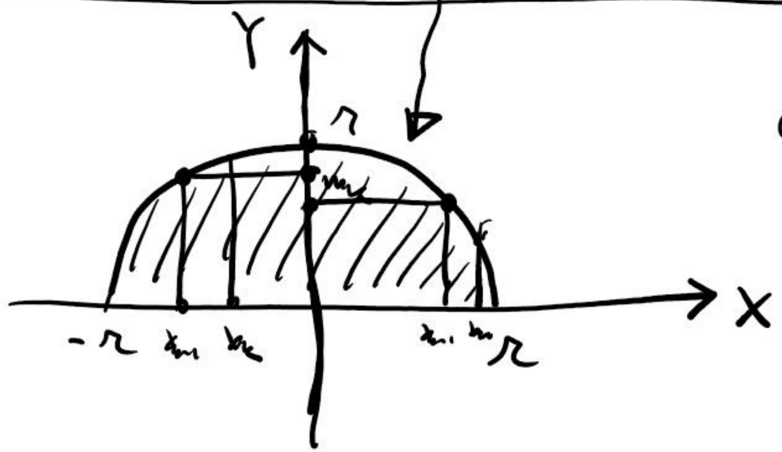
AVISO: 5ª Lista de exercícios deixa de ter C (parte sobre primitivas por partes)

A INTEGRAÇÃO:

3. Escrever as somas de DARBOUX da função

$$f(x) = \sqrt{r^2 - x^2} \quad \forall x \in [-r, r].$$

Discutir se f é integrável ou não em $[-r, r]$.



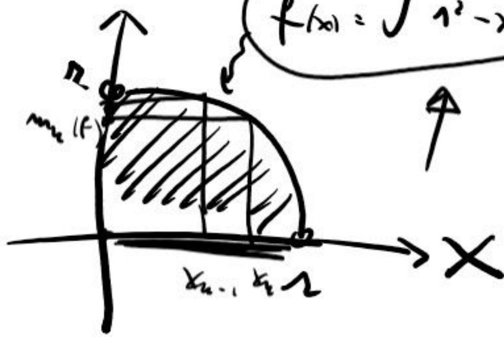
$$d = \{-r = x_0 < x_1 < \cdots < x_{n-1} < x_n = r\}$$

$$s_d(f) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1})$$

$$m_k(f) = \inf_{x \in [x_{k-1}, x_k]} f(x)$$

$x \in [x_{k-1}, x_k]$

$f(x) = \sqrt{1^2 - x^2}$



$$d = \{ 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = r \}$$

$$\to s_d(f) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}) = \sum_{k=1}^{n} \sqrt{r^2 - x_k^2}\,(x_k - x_{k-1})$$

$$\to S_d(f) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1}) = \sum_{k=1}^{n} \sqrt{r^2 - x_{k-1}^2}\,(x_k - x_{k-1})$$

B Primitiva Immediata:

(a) $$P\left(\frac{2}{\sqrt{x}} + \frac{1}{x^2} - 3\sqrt[5]{x^2}\right) = 2P\frac{1}{\sqrt{x}} + P\frac{1}{x^2} - 3P\sqrt[5]{x^2} = 2Px^{-\frac{1}{2}} + Px^{-2} - 3Px^{2/5}$$

$$= 2\frac{1}{-\frac{1}{2}+1}x^{-\frac{1}{2}+1} + \frac{1}{-2+1}x^{-2+1} - 3\cdot\frac{1}{\frac{2}{5}+1}x^{\frac{2}{5}+1} + c$$

$$= 4\sqrt{x} - \frac{1}{x} - \frac{15}{7}x^{7/5} + c$$

(b) $$P\, \frac{x^3-2x^2+3}{\sqrt{x}} = P\left(\frac{x^{6/2}}{x^{1/2}} - 2\,\frac{x^{4/2}}{x^{1/2}} + 3\,\frac{1}{x^{1/2}}\right) = P\left(x^{5/2} - 2\,x^{3/2} + 3\,x^{-1/2}\right) =$$

$$= Px^{5/2} - 2\,Px^{3/2} + 3\,Px^{-1/2} = \frac{1}{\frac{5}{2}+1}\,x^{\frac{5}{2}+1} - 2\,\frac{1}{\frac{3}{2}+1}\,x^{\frac{3}{2}+1} + 3\,\frac{1}{-\frac{1}{2}+1}\,x^{-\frac{1}{2}+1} + c$$

$$= \frac{2}{7}x^{7/2} - \frac{4}{5}x^{5/2} + 6\sqrt{x} + c$$

(c) $$P\, x\sqrt[4]{(x^2-1)^3} = P\, x\, \underbrace{(x^2-1)^{3/4}}_{u^\alpha \Rightarrow u' = 2x} = \frac{1}{2}\, P\, \underbrace{(2x)(x^2-1)^{3/4}}_{u' \cdot u^\alpha} =$$

$$= \frac{1}{2}\,\frac{1}{\frac{3}{4}+1}\,(x^2-1)^{\frac{3}{4}+1} = \frac{2}{7}(x^2-1)^{7/4} + c$$

(d) $$P\, \underbrace{x^3(x^2+1)^3}_{} = P\, x^3\left((x^2)^3 + 3(x^2)^2 + 3x^2 + 1\right) = P\, x^3\left(x^6 + 3x^4 + 3x^2 + 1\right)$$

$$= P(x^9+3x^7+3x^5+x^3) = Px^9 + 3Px^7 + 3Px^5 + Px^3 =$$

$$= \frac{1}{10}x^{10} + \frac{3}{8}x^8 + \frac{3}{6}x^6 + \frac{1}{4}x^4 + c$$

(e) $(a>0)$

$$P(a^x) = P e^{\log(a^x)} = P e^{x(\log a)} = \frac{1}{\log a} P \log a \, e^{x\log a} =$$

$$= \frac{1}{\log a} P \underbrace{(x\log a)' e^{x\log a}}_{u' \cdot e^u} = \frac{1}{\log a} e^{x\log a} + c$$

$$= \frac{a^x}{\log a} + c$$

(f) $$P \frac{x^2}{\sqrt{a^2+x^3}} = \frac{1}{3} P 3x^2 (a^2+x^3)^{-\frac{1}{2}} = \frac{1}{3} \frac{1}{-\frac{1}{2}+1} (a^2+x^3)^{-\frac{1}{2}+1} + c$$

$$= \frac{2}{3}\sqrt{a^2+x^3} + c$$

(g)

(g) $\int \frac{x^3}{1+x^4} = \frac{1}{4}\int \frac{4x^3}{1+x^4} = \frac{1}{4}\int \frac{(1+x^4)'}{1+x^4} = \frac{1}{4}\log(1+x^4)+c$

(h) $\int \frac{x}{1+x^4} = \frac{1}{2}\int \frac{\overset{u'}{2x}}{1+(x^2)^2} = \frac{1}{2}\operatorname{arctg}(x^2) + c$

$u = x^2$

$$\int \frac{u'}{1+u^2} = \operatorname{arctg} u + c$$

$u(x) = x^2$

$$\int \frac{x}{1+x^4} = \frac{1}{2}\int \frac{(2x)}{1+(x^2)^2} = \frac{1}{2}\int \frac{(x^2)'}{1+(x^2)^2} = \frac{1}{2}\operatorname{arctg}(x^2)+c$$

(i) $\int e^x\sqrt{2-e^x} = (-1)\int \underbrace{(-e^x)}_{u'}(\underbrace{2-e^x}_{u})^{\frac{1}{2}} \Rightarrow u' = -e^x \quad = -\frac{1}{\frac{1}{2}+1}(2-e^x)^{\frac{1}{2}+1}+c$

$$= -\frac{2}{3}\,(2-e^{x})^{3/2} + c$$

(j) $$\int \frac{1}{a^2+x^2} = \int \frac{1}{a^2\left(1+\left(\frac{x}{a}\right)^2\right)} = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a}\int \frac{1/a}{1+\left(\frac{x}{a}\right)^2}$$

$u \Rightarrow u' = \frac{1}{a}$

$$= \frac{1}{a}\,\operatorname{arctg}\left(\frac{x}{a}\right) + c$$

(k) $$\int \frac{1}{\sqrt{1-3x^2}} = \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}}{\sqrt{1-(x\sqrt{3})^2}} = \frac{1}{\sqrt{3}}\arcsin(x\sqrt{3}) + c$$

$u = x\sqrt{3} \Rightarrow u' = \sqrt{3}$

$$\int \frac{u'}{\sqrt{1-u^2}} = \arcsin u + c$$

(l) $$\int \frac{1}{\sqrt{a^2-x^2}} = \int \frac{1}{\sqrt{a^2\left(1-\left(\frac{x}{a}\right)^2\right)}} = \frac{1}{|a|}\int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

$u = \frac{x}{a} \Rightarrow u' = \frac{1}{a}$

$$= \frac{a}{|a|} \int \frac{1/a}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \frac{a}{|a|} \arcsin\left(\frac{x}{a}\right) + c$$

(m) $$\int \frac{1}{\sqrt{x^2+a^2}} = \int \frac{1}{\sqrt{a^2\left(\left(\frac{x}{a}\right)^2+1\right)}} = \frac{a}{|a|} \int \frac{1/a}{\sqrt{\left(\frac{x}{a}\right)^2+1}} =$$

$$= \frac{a}{|a|} \log\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2+1}\right) + c$$

(n) $$\int \frac{1}{\sqrt{x^2-a^2}} = \int \frac{1}{\sqrt{a^2\left(\left(\frac{x}{a}\right)^2-1\right)}} = \frac{a}{|a|} \int \frac{1/a}{\sqrt{\left(\frac{x}{a}\right)^2-1}} = \frac{a}{|a|} \log\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2-1}\right) + c$$

(o) $$\int \frac{1}{4+(x-3)^2} = \int \frac{1}{4\left(1+\left(\frac{x-3}{2}\right)^2\right)} = \frac{2}{4} \int \frac{1/2}{1+\underbrace{\left(\frac{x-3}{2}\right)}_{u}^2}$$

$$u \Rightarrow u' = \frac{1}{2}$$